

A remark concerning sinc integrals

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Abstract

We give a simple proof of Hanspeter Schmid's result that

$$K_n := 2 \int_0^\infty \cos t \prod_{k=0}^n \operatorname{sinc} \left(\frac{t}{2k+1} \right) dt = \frac{\pi}{2} \quad \text{if } n \in \{0, 1, \dots, 55\},$$

and $K_n < \pi/2$ if $n \geq 56$. Furthermore, we present two sinc integrals where the value $\pi/2$ is undercut as soon as $n \geq 418$ and $n \geq 3091$, respectively.

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1 Introduction

We define

$$\operatorname{sinc}(t) := \begin{cases} \frac{\sin t}{t} & \text{if } t \neq 0, \\ 1 & \text{if } t = 0. \end{cases}$$

Recently, Schmid [2] investigated the integrals

$$\tau_n := \int_0^\infty \prod_{k=0}^n \operatorname{sinc}(a_k t) dt \tag{1}$$

and

$$\varepsilon_n := \int_0^\infty 2 \cos a_0 t \prod_{k=0}^n \operatorname{sinc}(a_k t) dt \tag{2}$$

with positive real numbers $a_0 \geq a_1 \geq \dots \geq a_n$, and proved that

$$\left. \begin{aligned} \tau_0 = \frac{\pi}{2a_0}, \quad \tau_n = \frac{\pi}{2a_0} & \quad \text{for all } n \text{ with } \sum_{k=1}^n a_k \leq a_0, \\ \tau_n < \tau_{n-1} & \quad \text{for all other } n, \end{aligned} \right\} \tag{3}$$

and

$$\left. \begin{aligned} \varepsilon_0 = \frac{\pi}{2a_0}, \quad \varepsilon_n = \frac{\pi}{2a_0} & \quad \text{for all } n \text{ with } \sum_{k=1}^n a_k \leq 2a_0, \\ \varepsilon_n < \varepsilon_{n-1} & \quad \text{for all other } n. \end{aligned} \right\} \tag{4}$$

Result (4) was found by considering point-symmetric properties of the Fourier transforms

$$F_n(\omega) := \int_{-\infty}^{\infty} e^{-i\omega t} \prod_{k=0}^n \text{sinc}(a_k t) dt \quad (5)$$

(see [2, pp. 14-15, 17]).

A special case of the integral (1) is

$$J_n := \int_0^{\infty} \prod_{k=0}^n \text{sinc}\left(\frac{t}{2k+1}\right) dt. \quad (6)$$

Therefore, from (3) one easily finds [2, pp. 11-12, 16]

$$\left. \begin{aligned} J_n &= \pi/2 & \text{if } n &\in \{0, 1, \dots, 6\}, \\ J_n &< J_{n-1} & \text{if } n &\geq 7. \end{aligned} \right\} \quad (7)$$

This result was first proven by Borwein & Borwein [1, p. 86].

A special case of the integral (2) is

$$K_n := 2 \int_0^{\infty} \cos t \prod_{k=0}^n \text{sinc}\left(\frac{t}{2k+1}\right) dt. \quad (8)$$

From (4) one finds [2, pp. 12, 17]

$$\left. \begin{aligned} K_n &= \pi/2 & \text{if } n &\in \{0, 1, \dots, 55\}, \\ K_n &< K_{n-1} & \text{if } n &\geq 56. \end{aligned} \right\} \quad (9)$$

Due to the properties (7) and (9), Schmid called (6) and (8) as *curious integrals*.

2 Alternative proof of (4)

At first we show that the result (4) can be obtained in a very simple way without using the Fourier transforms (5) and its symmetry properties.

Using $2 \sin x \cos x = \sin 2x$, we have

$$\begin{aligned} \varepsilon_n &= \int_0^{\infty} 2 \cos(a_0 t) \prod_{k=0}^n \text{sinc}(a_k t) dt \\ &= 2 \int_0^{\infty} \cos(a_0 t) \text{sinc}(a_0 t) \prod_{k=1}^n \text{sinc}(a_k t) dt \\ &= 2 \int_0^{\infty} \cos(a_0 t) \frac{\sin(a_0 t)}{a_0 t} \prod_{k=1}^n \text{sinc}(a_k t) dt \end{aligned}$$

$$\begin{aligned}
&= 2 \int_0^\infty \frac{\sin(2a_0 t)}{2a_0 t} \prod_{k=1}^n \operatorname{sinc}(a_k t) \, dt \\
&= 2 \int_0^\infty \operatorname{sinc}(2a_0 t) \prod_{k=1}^n \operatorname{sinc}(a_k t) \, dt.
\end{aligned}$$

Substituting $b_0 = 2a_0$, and $b_k = a_k$ if $k \in \{1, 2, \dots, n\}$, we have

$$\varepsilon_n = 2 \int_0^\infty \prod_{k=0}^n \operatorname{sinc}(b_k t) \, dt.$$

From (3) it follows that

$$\left. \begin{aligned}
\varepsilon_0 &= 2 \cdot \frac{\pi}{2b_0} = \frac{\pi}{b_0}, \quad \varepsilon_n = 2 \cdot \frac{\pi}{2b_0} = \frac{\pi}{b_0} \quad \text{for all } n \text{ with } \sum_{k=1}^n b_k \leq b_0, \\
\varepsilon_n &< \varepsilon_{n-1} \quad \text{for all other } n.
\end{aligned} \right\} \quad (10)$$

Substituting back, (4) follows immediately.

This shows that integral (2) may be considered as special case of integral (1), and result (4) as special case of result (3).

3 Proof of (9)

From Section 2 we find that integral (8) may be written as

$$K_n = 2 \int_0^\infty \prod_{k=0}^n \operatorname{sinc}(b_k t) \, dt$$

where

$$b_0 := 2, \quad b_k := \frac{1}{2k+1} \quad \text{if } k \in \{1, 2, \dots, n\}.$$

Using (10) yields

$$\begin{aligned}
K_0 &= \frac{\pi}{2}, \quad K_n = \frac{\pi}{2} \quad \text{for all } n \text{ with } \sum_{k=1}^n \frac{1}{2k+1} \leq 2, \\
K_n &< K_{n-1} \quad \text{for all other } n.
\end{aligned}$$

Result (9) follows.

4 Many other curious integrals

We put

$$I_n(b) := b \int_0^\infty \operatorname{sinc}(bt) \prod_{k=0}^n \operatorname{sinc}\left(\frac{t}{2k+1}\right) \, dt$$

with a real number $b \geq 1$. From (3) it is obviously clear that

$$b \int_0^\infty \operatorname{sinc}(bt) \, dt = \frac{\pi}{2},$$

and

$$I_n(b) = \frac{\pi}{2} \quad \text{if} \quad \sum_{k=0}^n \frac{1}{2k+1} \leq b,$$

$$I_n(b) < \frac{\pi}{2} \quad \text{if} \quad \sum_{k=0}^n \frac{1}{2k+1} > b.$$

Here are some results for special cases:

- $I_n(1) = \pi/2$ if $n = 0$, $I_n(1) < \pi/2$ if $n \geq 1$,
- $I_n(2) = \pi/2$ if $n \in \{0, 1, \dots, 6\}$, $I_n(2) < \pi/2$ if $n \geq 7$,
- $I_n(3) = \pi/2$ if $n \in \{0, 1, \dots, 55\}$, $I_n(3) < \pi/2$ if $n \geq 56$,
- $I_n(4) = \pi/2$ if $n \in \{0, 1, \dots, 417\}$, $I_n(4) < \pi/2$ if $n \geq 418$,
- $I_n(5) = \pi/2$ if $n \in \{0, 1, \dots, 3090\}$, $I_n(5) < \pi/2$ if $n \geq 3091$.

A calculation with Mathematica gave

$$I_7(2) = \frac{168579263752211300739165075916829279}{337158527504429357358419617830000000} \pi$$

$$\approx 0.49999999999998998115 \pi.$$

References

- [1] D. Borwein, J. M. Borwein: Some remarkable properties of sinc and related integrals, *The Ramanujan Journal* **5** (2001), 73-89.
- [2] H. Schmid: Two curious integrals and a graphic proof, *Elem. Math.* **69** (2014), 11-17.

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